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APPLICATION OF HIGHER ORDER THEORIES TO THE BENDING ANALYSIS OF LAYERED COMPOSITE PLATES

K. ROHWER
DLR, Institute for Structural Mechanics, Postfach 3267, D-3200 Braunschweig, Germany

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Abstract—An increasing number of higher order theories for the analysis of laminated plates has been published recently. However, there is as yet a lack of rigorous information about a sensible range of application. To help remedy this deficiency, seven different theories with displacement functions valid for the complete plate thickness are compared against the exact three-dimensional elasticity solution. Rectangular plates with varying slenderness ratios, layer numbers and thicknesses, edge ratios, and material property relations are examined. Comparing their displacements and stress distributions reveals application limits for the classical lamination theory as well as advantages and disadvantages of the respective higher order theories. Therewith, guiding rules evolve for a reliable and efficient bending analysis of layered composite plates.

1. INTRODUCTION

Plates out of fiber-reinforced materials have been built and analyzed for many years. The basis for the computations was, and still is in most cases, the so-called classical lamination theory (CLT), a straightforward extension of the Kirchhoff (1850) plate theory towards layered anisotropic material. Both theories neglect transverse shear strains, the effect of which is known to be small for slender plates. Fiber-reinforced material, however, is more susceptible to transverse shear than its isotropic counterpart, thus reducing the range of applicability of CLT. Adopting Mindlin's (1951) theory, Whitney and Pagano (1970) relaxed this restriction. But the advancement had to be paid for by two additional functional degrees of freedom: the shear rotations. Nevertheless, the theory is very popular with finite element developers because it needs C^0-continuous functions only. Unfortunately, there is still some uncertainty concerning suitable transverse shear stiffnesses of layered composite plates.

Recently, an increasing number of "higher order" theories for layered plates has been published. The term indicates that the displacement distribution over the plate thickness is represented by polynomials of higher than first order. In general, a higher approximation will lead to better results but also requires a more expensive computational effort. There is as yet little information about advantages and disadvantages of the various approaches; a potential user has hardly any criterion to decide which theory is most suitable to solve this problem.

In order to provide some insight into the range of applicability and the power of the various theories, computational results will be presented. The study will be confined to theories with continuous displacement functions across the complete plate thickness; layer-wise approximations are not considered. Bending and transverse shear of symmetrically stacked plates will be treated. The exact three-dimensional (3-D) elasticity solution after Pagano (1970) will be applied as a yardstick. Since this solution is available only under certain conditions the test problem cannot be arbitrary. It includes, however, varying slenderness ratios, material property relations, edge ratios and layer numbers. Therewith, it will be general enough so that a comparison with the corresponding results obtained from the different plate theories can yield useful information about their scope.

2. SPECIFICATIONS OF EXAMINED PLATE THEORIES

Plates are defined as structural components with a thickness much smaller than the in-plane dimensions. That allows one to separate the dependency on the thickness direction and to assume distribution functions for the displacements. It leads to two-dimensional theories which reduces the computational effort required for the solution. A general formulation based on this procedure is presented by Reddy (1987). The higher order theories can be looked upon as special cases thereof. But in order to elucidate the differences it is necessary to describe the assumptions in more detail.

As a common and useful practice, the plate behavior will be described in the frame of a Cartesian coordinate system (x, y, z), where the x- and y-axis specify the reference plane and the z-axis runs in the thickness direction. Displacement components in x-, y- and z-directions are named u, v and w, respectively (cf. Fig. 1).

Constant in-plane displacements would describe membrane deformations which are not considered here. To allow for bending deformations the most simple approach is

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ w_0 \end{Bmatrix} + \begin{Bmatrix} u_1 \\ v_1 \\ 0 \end{Bmatrix} \cdot z. \tag{1}$$

It forms the basis for the Whitney–Pagano (1970) theory as well as for the CLT. Neglecting the normal stresses in z-direction and eliminating the corresponding strains changes the definition of the bending stiffnesses. Only the treatment of transverse shear is different between the two theories. CLT completely neglects transverse shear strains, which relates u_1 and v_1 to the derivatives of w:

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = -\begin{Bmatrix} \partial w_0/\partial x \\ \partial w_0/\partial y \end{Bmatrix}. \tag{2}$$

Therewith, the only functional degree of freedom left is w_0. Whitney and Pagano (1970) allowed for transverse shear strains but it is difficult to define the corresponding stiffnesses. The integral of the shear modulus over the thickness must be reduced because of non-constant shear stresses. Several attempts were made to derive generally valid reduction factors, but Wittrick (1987) proved that for orthotropic material it is impossible to choose effective shear moduli independent of the displacement mode. Following a proposal by Lehar (1984) the author (Rohwer, 1988) has assumed two cylindrical bending modes to set up a procedure which provides improved values. In the following these values are used exclusively when applying the Whitney–Pagano theory.

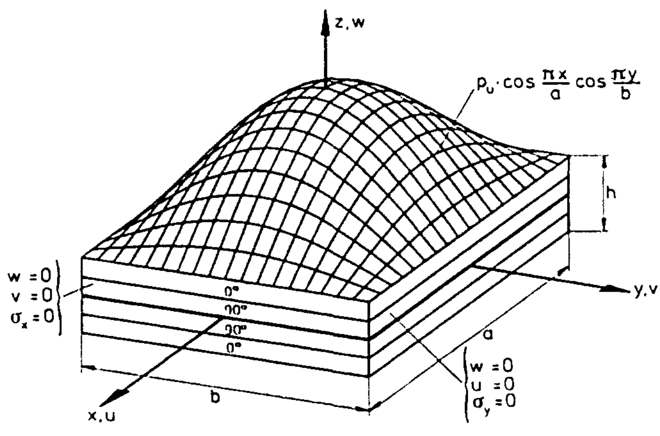

Fig. 1. Test plate configuration.

Increasing the in-plane displacement approximation order to quadratic polynomials would not make much sense, because the condition of vanishing transverse shear strains at the upper and lower plate surface eliminates the quadratic components. Therefore, the simplest higher order theories along this line are based on

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ w_0 \end{Bmatrix} + \begin{Bmatrix} u_1 \\ v_1 \\ 0 \end{Bmatrix} \cdot z + \begin{Bmatrix} u_3 \\ v_3 \\ 0 \end{Bmatrix} \cdot z^3. \tag{3}$$

A reduction of the number of functional degrees of freedom is desirable, but the way it is performed differs between the different authors.

Murthy (1981) introduced new variables which he obtained by means of averaging in-plane displacements and rotations through the thickness with the aid of a least-square approximation. Thus he defined mean rotations

$$\begin{Bmatrix} \beta_x \\ \beta_y \end{Bmatrix} = \frac{12}{h^3} \int_{-h/2}^{h/2} \begin{Bmatrix} u \\ v \end{Bmatrix} \cdot z \, dz, \tag{4}$$

which are related to the initial functions by

$$\begin{Bmatrix} \beta_x \\ \beta_y \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} + \frac{3h^2}{20} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}. \tag{5}$$

The final distribution over z of the in-plane displacements then reads

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \left(\frac{5}{4} z - \frac{5}{3h^2} z^3\right) \begin{Bmatrix} \beta_x \\ \beta_y \end{Bmatrix} + \left(\frac{1}{4} z - \frac{5}{3h^2} z^3\right) \begin{Bmatrix} \partial w_0/\partial x \\ \partial w_0/\partial y \end{Bmatrix}. \tag{6}$$

Transverse shear strains based on these displacement functions are more realistic than in the case of the Whitney–Pagano theory. Accordingly, a correction factor for the corresponding stiffnesses is no longer necessary. Murthy used the same equilibrium equations as Whitney–Pagano. Due to the definition of mean rotations, however, they cannot be derived via the principle of virtual displacements.

Reddy (1984) used the condition of vanishing transverse shear strains at the upper and lower surface not only to eliminate the quadratic displacement components but also to replace u_3 and v_3:

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = -\frac{4}{3h^2} \begin{Bmatrix} u_1 + \partial w_0/\partial x \\ v_1 + \partial w_0/\partial y \end{Bmatrix}. \tag{7}$$

The distribution over z of the in-plane displacements then reads

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = z \cdot \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} - \frac{4}{3h^2} z^3 \begin{Bmatrix} u_1 + \partial w_0/\partial x \\ v_1 + \partial w_0/\partial y \end{Bmatrix}. \tag{8}$$

Linear and cubic terms of the displacements lead to two different types of bending and torsional strains; they work at two different types of bending and torsional moments. And there are also two different types of transverse shear strains and transverse shear forces.

Based on Reddy's theory, Senthilnathan *et al.* (1987) introduced a further reduction of the number of functional degrees of freedom by splitting up the transverse displacements w_0 into a bending and a shear contribution,

$$w_0 = w_b + w_s. \tag{9}$$

The rotations are then identified with the derivatives of w_b as follows:

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = -\begin{Bmatrix} \partial w_b/\partial x \\ \partial w_b/\partial y \end{Bmatrix}. \tag{10}$$

Therewith, the displacement distribution over z reads

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ w_b + w_s \end{Bmatrix} - z \cdot \begin{Bmatrix} \partial w_b/\partial x \\ \partial w_b/\partial y \\ 0 \end{Bmatrix} - \frac{4}{3h^2} z^3 \begin{Bmatrix} \partial w_s/\partial x \\ \partial w_s/\partial y \\ 0 \end{Bmatrix}. \tag{11}$$

As with Reddy's theory there are still two different types of bending and torsional strains as well as transverse shear strains. The latter now depend on the contribution w_s alone.

Other possibilities for a higher order theory evolve from a displacement approximation of the type

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ w_0 \end{Bmatrix} + \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} \cdot z + \begin{Bmatrix} 0 \\ 0 \\ w_2 \end{Bmatrix} \cdot z^2. \tag{12}$$

Such an approach was introduced by Whitney and Sun (1974) who still used transverse shear reduction factors. Kwon and Akin (1987) pointed out that vanishing shear strains at $z = \pm h/2$ requires w_1 to be zero. Furthermore, they used the same condition to replace the derivatives of the quadratic component w_2 by

$$\begin{Bmatrix} \partial w_2/\partial y \\ \partial w_2/\partial x \end{Bmatrix} = -\frac{4}{3h^2} \begin{Bmatrix} v_1 + \partial w_0/\partial y \\ u_1 + \partial w_0/\partial x \end{Bmatrix}. \tag{13}$$

The remaining functions, w_0, u_1, v_1, are identical to those from the Whitney-Pagano theory. Transverse shear stiffnesses, however, are directly obtained; no reduction factor must be introduced.

Reissner (1975) introduced a combination of cubic in-plane and quadratic transverse displacement approximations. It was applied to laminated plates by Lo *et al.* (1977) and further developed by Pandya and Kant (1988):

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ w_0 \end{Bmatrix} + \begin{Bmatrix} u_1 \\ v_1 \\ 0 \end{Bmatrix} \cdot z + \begin{Bmatrix} 0 \\ 0 \\ w_2 \end{Bmatrix} \cdot z^2 + \begin{Bmatrix} u_3 \\ v_3 \\ 0 \end{Bmatrix} \cdot z^3. \tag{14}$$

The development can be extended further along this line whereby the number of functional degrees of freedom is increased. However, since that number is of major influence on the necessary computational effort it should be kept as low as possible.

Displacement approximations form the basis for the cited theories. But engineering applications also require information about the stresses. The following comparison will be confined to in-plane and transverse shear stresses. Transverse normal stresses are considered small and therefore explicitly set to zero in most plate theories. That excludes examples where the free-edge effect is of importance.

In-plane stresses σ_x, σ_y and τ_{xy} are in all cases obtained by multiplying the corresponding strains with the in-plane stiffnesses. These stiffnesses may change from layer to layer resulting in discontinuous stress functions over the plate thickness. It is noteworthy that

Table 1. Examined plate theories

ID	Theory/author	Functions
C1	CLT: Kirchhoff (1850)	w_0
S2	Senthilnathan *et al.* (1987)	w_b, w_s
W3	Whitney–Pagano (1970)	w_0, u_1, v_1
M3	Murthy (1981)	w_0, β_x, β_y
R3	Reddy (1984)	w_0, u_1, v_1
K3	Kwon–Akin (1987)	w_0, u_1, v_1
P6	Pandya–Kant (1988)	w_0, w_2, u_1, v_1, u_3, v_3
3D	Pagano (1970)	w, u, v

most theories require stiffnesses derived with the aid of a plane stress assumption. The Pandya–Kant theory, however, uses the original three-dimensional constants.

With the exception of the CLT the transverse shear stresses could also be determined via the material law. However, better and more comparable results can be expected from an application of the equilibrium conditions:

$$\tau_{xz} = -\int\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}\right)\mathrm{d}z,$$

$$\tau_{yz} = -\int\left(\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x}\right)\mathrm{d}z. \tag{15}$$

Derivatives of the in-plane stresses are to be integrated over the thickness. This procedure will be applied to all plate theories. It allows us to obtain transverse shear stresses even for the CLT.

Test calculations are performed with all theories listed in Table 1. The identifier (ID) specified in column 1 indicates an author's name or a theory, and the number of functional degrees of freedom involved. It should be pointed out that in many cases the theories have been developed by several different authors; the reference cited here may serve as an indication.

3. TEST PROBLEM DEFINITION

For the intended bending analysis a layered composite plate with rectangular ground view is considered a suitable test case. Figure 1 shows the configuration. The edge lengths in x- and y-directions are a and b, respectively; the total thickness is h. Only symmetric stacking sequences are taken into account, with $z = 0$ forming the symmetry plane. The angle of the fiber reinforcement direction is measured as a positive rotation around the z-coordinate with the x-axis being the 0°-direction. As a major restriction only 0°- and 90°-layers can be applied; for fiber angles other than these no closed-form 3-D elasticity solution exists.

Boundary conditions along the edges are some kind of simple support with prevented displacements in both tangential directions and no stresses in the normal direction. With the condition $w = 0$ prescribed for the whole edge surface, the well-known edge effect cannot develop. At the upper and lower plate surface normal loads are applied; the transverse shear stresses are assumed to vanish, whereas the normal stresses must equalize the load. In order to simplify the solution procedure the load is chosen to be distributed in a double cosine manner. That allows a one-term approximation for every displacement function. Furthermore, the stresses, too, are represented by single functions in the in-plane coordinates which excludes stress concentration problems. Table 2 sums up the boundary conditions. Interface conditions between the layers (k) and ($k+1$) are

$$w^{(k)} = w^{(k+1)},\quad u^{(k)} = u^{(k+1)},\quad v^{(k)} = v^{(k+1)},$$

$$\sigma_z^{(k)} = \sigma_z^{(k+1)},\quad \tau_{xz}^{(k)} = \tau_{xz}^{(k+1)},\quad \tau_{yz}^{(k)} = \tau_{yz}^{(k+1)}.$$

Table 2. Boundary conditions

Boundary	Condition
$x = \pm a/2$	$w = 0,\quad v = 0,\quad \sigma_x = 0$
$y = \pm b/2$	$w = 0,\quad u = 0,\quad \sigma_y = 0$
$z = -h/2$	$\tau_{xz} = 0,\quad \tau_{yz} = 0,\quad \sigma_z = -p_l \cdot \cos(\pi x/a) \cdot \cos(\pi y/b)$
$z = +h/2$	$\tau_{xz} = 0,\quad \tau_{yz} = 0,\quad \sigma_z = +p_u \cdot \cos(\pi x/a) \cdot \cos(\pi y/b)$

Therewith, the test problem is sufficiently defined. Actual dimensions and material properties remain variable.

Pagano (1970) has shown that this type of problem can be solved exactly within the scope of 3-D elasticity theory. For each layer the boundary conditions along the edges are satisfied if trigonometric functions of x and y are assumed for the three displacements. In z-direction exponential functions must be applied, leading to an eigenvalue problem of sixth order. The unknown coefficients of the complete solution are determined by interface and boundary conditions at the upper and lower surface, respectively. That specifies a linear equation system of $6n$ equations, where n is the number of layers involved.

4. COMPUTATIONAL RESULTS

4.1. *Slenderness effect*

It is obvious that the Kirchhoff assumptions hold for very slender plates. If the thickness is small enough, away from the edges there is no room for developing considerable transverse shear stresses between upper and lower surfaces, where they must vanish because of equilibrium conditions. Transverse normal stresses must exist at least to equilibrate the load. But their strain energy contribution and therewith their influence on the global displacements must remain small. Finally, the distribution of in-plane displacements will tend to be linear; higher order distributions would result in self-equilibrating stresses which die out rather soon in slender plates.

With decreasing slenderness ratios, however, these arguments gradually lose their validity. Effects which are neglected in the classical lamination theory will obtain a growing influence. Depending on the degree of approximation some of these effects are taken care of by the higher order theories mentioned above. But all higher order theories suffer from the increased effort required for the solution. Therefore, as long as the classical lamination theory yields satisfactory results it is the first choice.

Information about the range of applicability of the different theories with respect to the slenderness ratio will be gathered by means of the example specified above. In the first instance a stacking sequence of $[0, 90]_s$ and material properties as listed in Table 3 are chosen. They represent a conventional carbon HT-fiber reinforced plastic (CFRP). The plate thickness is kept to unity while the edge lengths are varying with a fixed ratio of

$$a/b = \sqrt[4]{D_{xx}/D_{yy}}, \tag{16}$$

where D_{xx} and D_{yy} are the bending stiffnesses in x- and y-direction, respectively. Rather than unity, this ratio was chosen so that both directions participate equally in carrying the load. For the 3-D solution half of the load is applied at the upper and the lower surfaces, respectively. This has no significant influence on the deflection of the reference surface, but results in an anti-symmetric displacement mode.

Table 3. Material properties for slenderness test

$E_L =$	138.0 kN mm^{-2}
$E_T =$	9.3 kN mm^{-2}
$G_{LT} =$	4.6 kN mm^{-2}
$\nu_{LT} =$	0.3
$\nu_{TT} =$	0.5

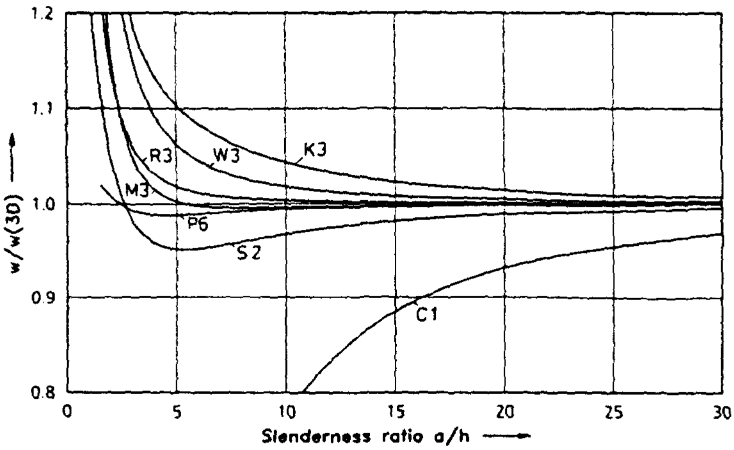

Fig. 2. Center deflection of $[0,90]_s$ CFRP plates, $h = [0.25, 0.25]_s$.

As the main indicator for adequacy of the respective theory the center plate deflection is calculated and depicted in Fig. 2. For a varying slenderness ratio the calculated center deflection is related to the corresponding value obtained in the 3-D analysis. The deviation from unity, therefore, represents the error inherent in the approximate theory. As could be expected, the classical lamination theory (C1) delivers too small deflections. The deviation increases with reducing slenderness. That can be traced back to the neglect of transverse shear. For slenderness ratios less than 25 the errors exceed 5%, which in some cases may not be tolerable so that a higher order theory is recommended.

Initial choice for an improvement usually is to include transverse shear effects. With Whitney-Pagano's theory (W3), using transverse shear stiffnesses as proposed by the present author, the deflections are much better. The fact that they are overestimated may be due to the equilibrium approach applied in the shear stiffness development. Excellent accuracy is obtained with the theories of Murthy (M3) and Reddy (R3), whereas Kwon and Akin's (K3) is inferior. This can be explained by the cubic approximation of in-plane displacements used by Murthy as well as Reddy, while Kwon and Akin proposed a linear distribution only. Reasonably good are the results obtained with the theory by Senthilnathan *et al.* (S2), bearing in mind the reduced effort required for the only two functional degrees of freedom. Best accuracy even at slenderness ratios as low as 2.0 is reached following Pandya and Kant (P6), but that has to be paid for by using six functional degrees of freedom. Generally it can be stated from Fig. 2 that for slenderness ratios smaller than 5 a plate theory is hardly suitable to treat the given example.

Besides the center deflection it is the distribution of the in-plane displacements and stresses over the thickness which indicates the accuracy of the considered plate theory. Therefore, these functions will be compared with those obtained from the 3-D theory. Maximum values of stresses and displacements will appear at different plate locations. They are put together in Table 4. To make the differences in the in-plane displacements obtained with the various theories more visible, the corresponding values of the classical lamination theory are subtracted, thereby eliminating a global rotation of the cross-section. The remainder is then scaled using the respective maximum displacement of the classical lamination theory. The stresses are scaled with p, the maximum of the load function, and with certain slenderness ratios.

Table 4. Locations of maximum displacements and stresses

Component	Location	
u, τ_{xz}	$x = \pm a/2$,	$y = 0$
v, τ_{yz}	$x = 0$,	$y = \pm b/2$
σ_x, σ_y	$x = 0$,	$y = 0$
τ_{xy}	$x = \pm a/2$,	$y = \pm b/2$

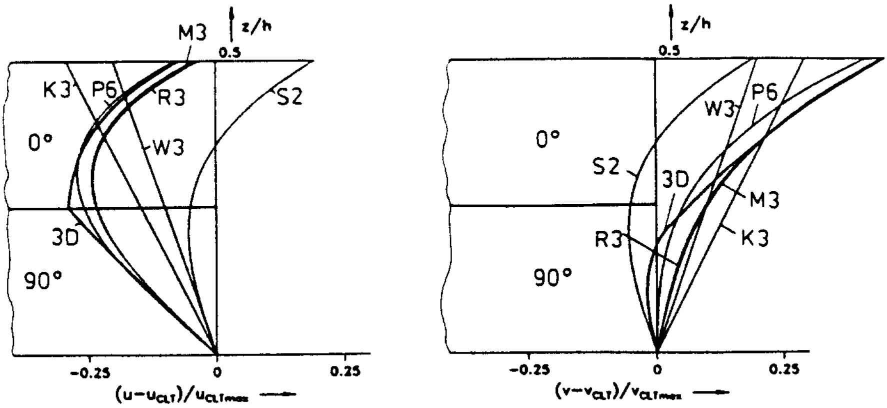

Fig. 3. In-plane displacement distribution, $a/h = 5.0$.

Figure 3 presents the in-plane displacements for a slenderness ratio of 5. Only the upper half of the cross-section is depicted; in the lower half the displacement distribution is anti-symmetric, carrying no further information. That also holds for the 3-D solution, which is marked by the thick line. Conspicuous is its pronounced nonlinearity which, in case of the u-displacements, even shows a kink at the layer interface. This certainly cannot be modeled by the polynomials of the plate theories. The cubic functions as applied by Murthy (M3), Reddy (R3) and Pandya and Kant (P6) have a better chance to approximate this behavior than the linear functions of the Whitney–Pagano theory (W3) or the Kwon and Akin approach (K3). But the latter two at least represent some kind of mean shear rotation. Unexpectedly large deviations result from the theory proposed by Senthilnathan *et al.* (S2). One of the two functional degrees of freedom available in this theory is used to represent the Kirchhoff rotations, which are subtracted in the drawing. Obviously, the remaining one function is not sufficient to treat layered orthotropic material in a satisfactory manner.

Using the same scale, Fig. 4 shows the relations at a slenderness ratio of 10. It makes clear that the deviation from the classical lamination theory vanishes quickly. For a slenderness ratio of 25, where the transverse displacements are still some 4% off, the difference in in-plane displacements between 3-D and the plate theories are within the drawing accuracy.

The corresponding stresses are given in Figs 5 and 6. Again, only the upper half of the laminate is depicted. Symmetry conditions require anti-symmetry of the stresses σ_x, σ_y and τ_{xy}, whereas the transverse shear stresses τ_{xz} and τ_{yz} must be symmetric. Since the total stresses rather than differences are plotted here, the discrepancies between the theories do not look so spectacular. The normal stress distributions show the expected discontinuity

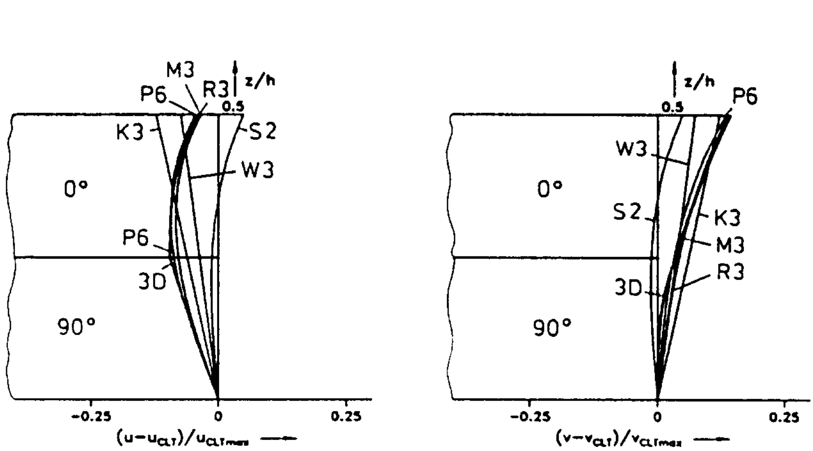

Fig. 4. In-plane displacement distribution, $a/h = 10.0$.

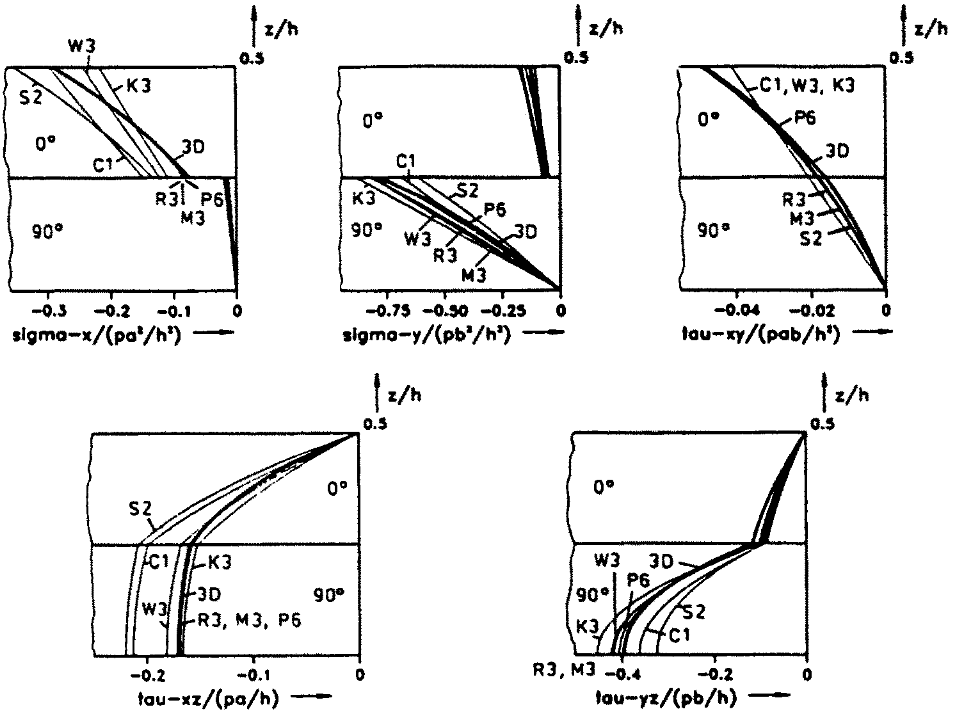

Fig. 5. Stress distribution, $a/h = 5.0$.

due to the change in the material properties. In-plane shear stiffnesses are equal for 0°- and 90°-layers, so that the τ_{xy}-functions are continuous. Transverse shear stiffnesses of the 0°-layer, however, differ from those of the 90°-layer due to the difference between ν_{LT} and ν_{TT}. That leads to the slope discontinuity; the functions themselves are continuous because of the enforced equilibrium conditions.

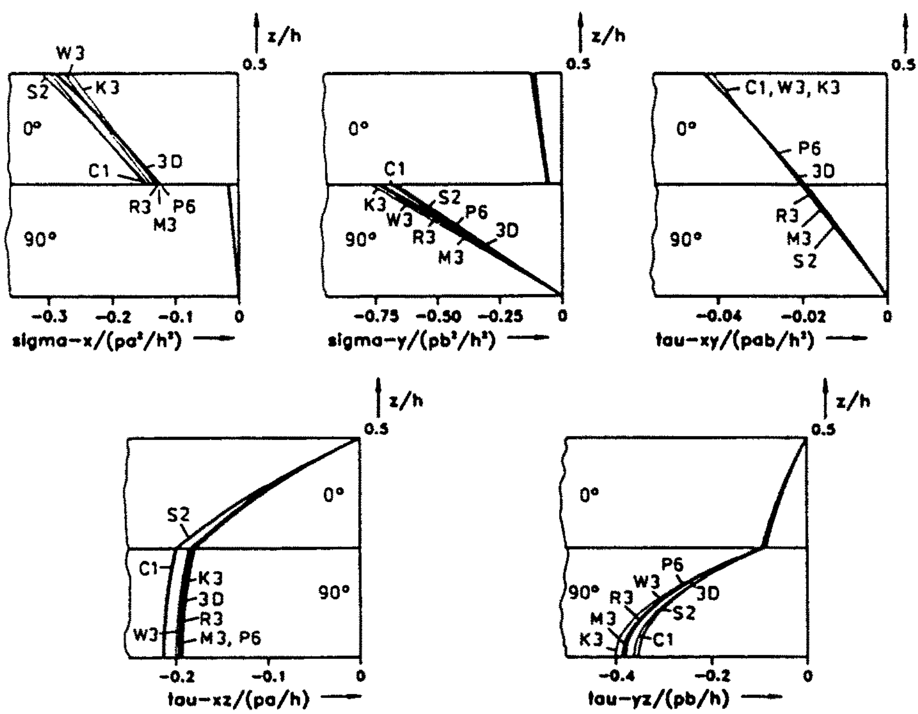

Fig. 6. Stress distribution, $a/h = 10.0$.

Figure 5 shows the stresses at a slenderness ratio of 5. With their linear approximations the CLT, the Whitney–Pagano theory and Kwon and Akin's approach have difficulties in modeling the normal stresses, especially σ_x in the 0°-layer. This is much better with the other theories, except that of Senthilnathan *et al.* (S2). As could be expected by comparison with the in-plane displacements, the in-plane normal stresses in an orthotropic material cannot be modeled in a satisfactory manner with one function only. In the case of in-plane shear stresses all theories render acceptable results. It is noteworthy that the linear approximation theories (C1, W3, K3) give identical stresses τ_{xy} though their in-plane displacement distributions are different. The transverse shear stresses τ_{xz} obtained with C1 and S2 are somewhat high. In turn, the corresponding values for τ_{yz} must be low to satisfy equilibrium. With the other theories the results are remarkably close to the exact 3-D solution, especially if the relatively low slenderness is considered. The accuracy of all stress components increases with increasing slenderness. At a ratio of 10, already, the differences can hardly be displayed, as Fig. 6 shows.

The influence of unequal layer thicknesses is studied by means of a [0, 90], laminate with a thickness of h = [0.40, 0.10],. Transverse deflections are depicted in Fig. 7. As compared with Fig. 2 the results obtained with Reddy's (R3), Murthy's (M3), and Kwon and Akin's (K3) theories are now worse. Also, Senthilnathan's (S2) theory delivers larger deviations from the 3-D results for slenderness ratios above 5. Only the curves marked W3 and P6 show equal or even smaller errors than before. These findings are confirmed by the graphs of the in-plane displacement distributions given in Fig. 8. As in Figs 3 and 4 Senthilnathan's theory (S2) yields equal functions for $(u - u_{CLT})/u_{CLT_{max}}$ and $(v - v_{CLT})/v_{CLT_{max}}$. That explains its difficulties in approximating the 3-D distribution. The same tendency appears with the stresses which are given in Fig. 9. In particular the results obtained with

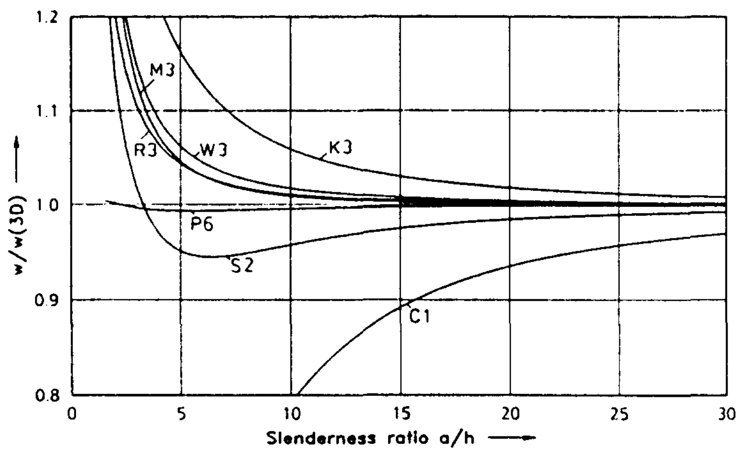

Fig. 7. Center deflection of [0, 90], CFRP plates, h = [0.40, 0.10],.

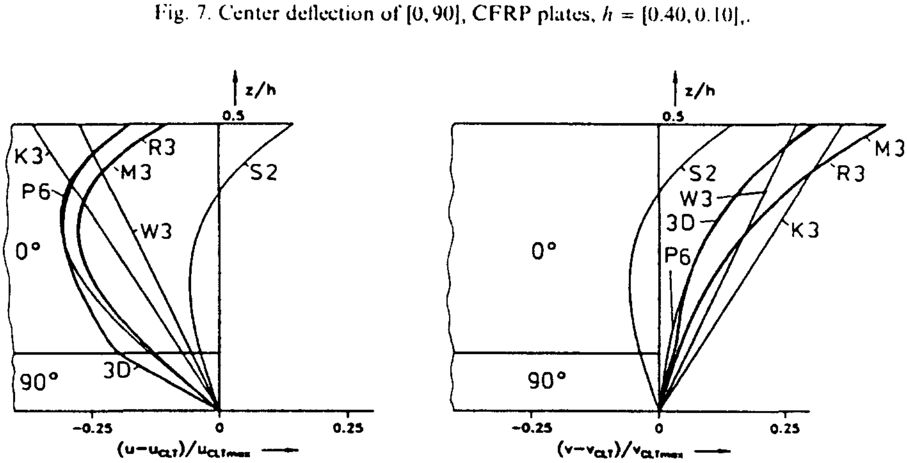

Fig. 8. In-plane displacement distribution, a/h = 5.0.

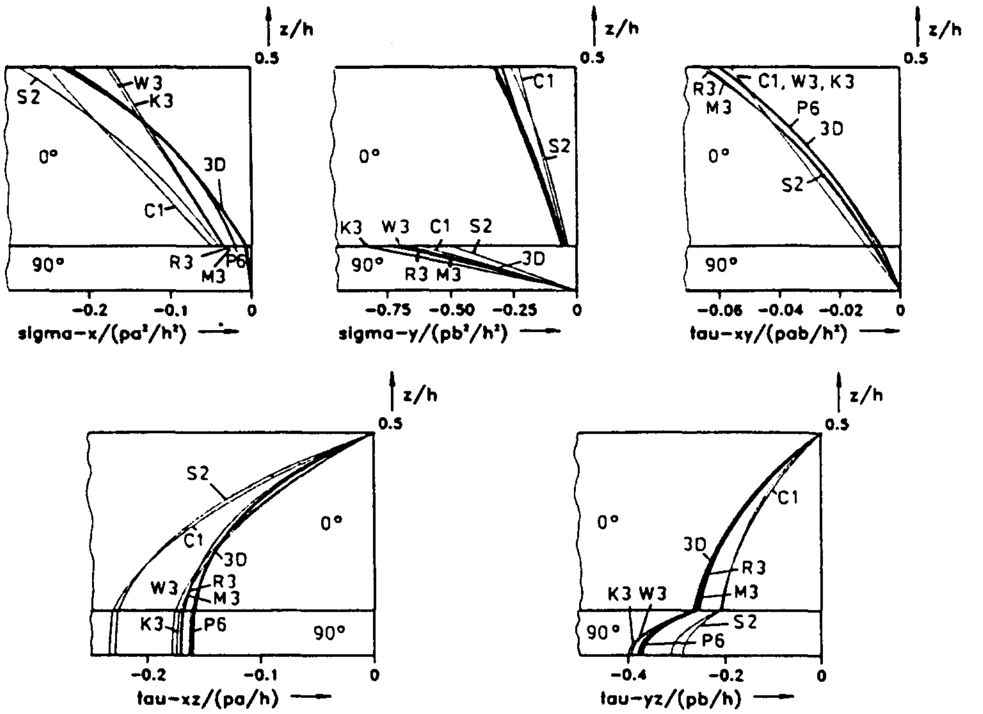

Fig. 9. Stress distribution, $a/h = 5.0$.

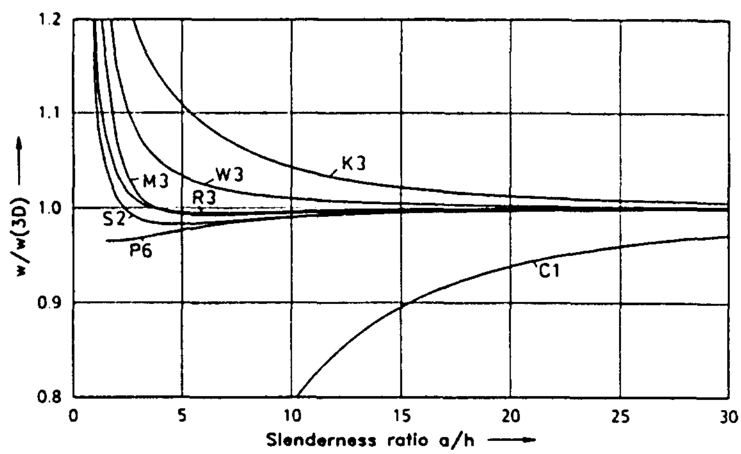

Fig. 10. Center deflection of [0, 90, 0, 90], CFRP plates, $h = [0.125]_8$.

CLT and Senthilnathan's theory now deviate somewhat more from the 3-D values as compared with Fig. 5.

Laminates with four layers only are not very common; usually the number is much larger. In order to study the influence of the layer number a [0, 90, 0, 90], laminate with a thickness of $[0.125]_8$ was analyzed. Figure 10 shows the transverse deflections. Results obtained with the classical lamination theory are slightly improved, whereas the theory of Pandya and Kant (P6) is not as good as in the previous cases, especially for low slenderness ratios. Senthilnathan's theory (S2), however, gives excellent results. The reason becomes obvious when inspecting the in-plane displacement distribution in Fig. 11. For a slenderness ratio of 5, both u- and v-components show the zig-zagging mode as reported already by Pagano and Hatfield (1972). It is difficult to model such a mode with low order polynomials defined over the whole cross section. However, since the transverse shear effect is not very different in the x- and y-directions, the one function w_s of Senthilnathan's theory is sufficient to yield satisfying approximations. Besides, the zig-zagging dies out rapidly with increasing

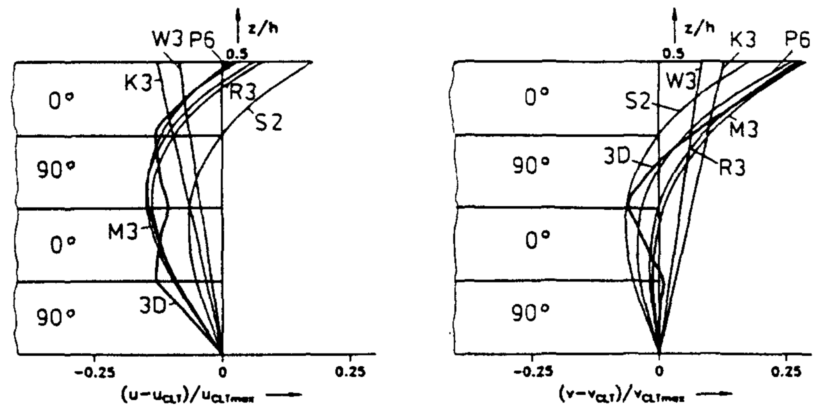

Fig. 11. In-plane displacement distribution, $a/h = 5.0$.

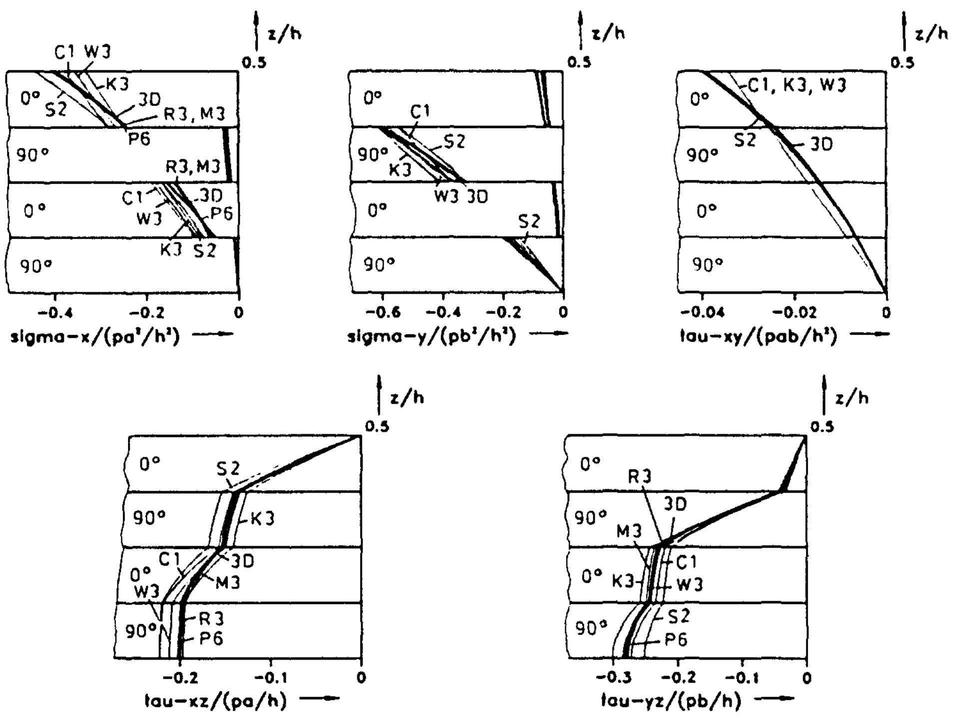

Fig. 12. Stress distribution, $a/h = 5.0$.

slenderness. At $a/h = 10$, which can still be considered a thick plate, the distribution of the in-plane displacements is already relatively smooth.

Stress distributions, too, have improved, as a comparison between Figs 5 and 12 shows. The in-plane normal stresses σ_x in the 0°-layers are still somewhat off, but notably both transverse shear stresses are now quite close to the 3-D solution. With only minor deviations in some cases that holds for all theories. Even the CLT gives good transverse shear stresses if determined via the equilibrium conditions (eqn (15)). An increasing number of layers obviously reduces the errors that occur when calculating stresses from lower order theories.

4.2. *Edge ratio effect*

So far the edge ratio has been fixed according to eqn (16). The question arises, how important the influence of this parameter on the accuracy of the different theories might be. Some light can be shed on this matter by analyzing $[0, 90]_r$ CFRP plates with equal layer thickness and a fixed slenderness ratio of $a/h = 15$. The material properties are kept as

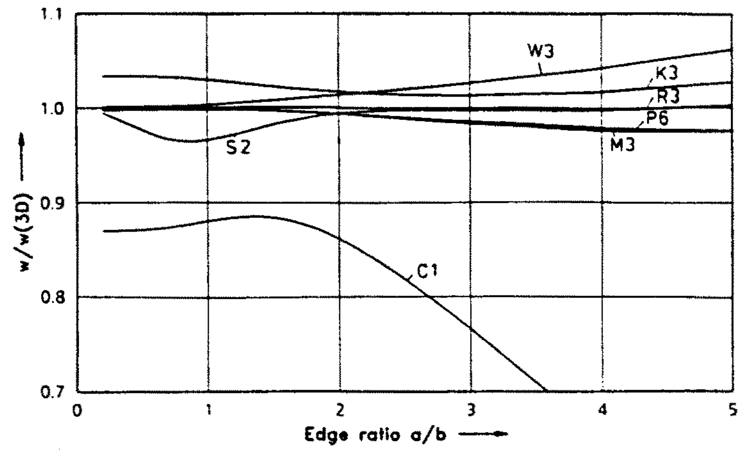

Fig. 13. Center deflection of [0, 90], CFRP plates. a/h = 15.0.

before (Table 3), but the edge ratio a/b is varied between 0.2 and 5.0. Figure 13 shows the results.

For edge ratios between 0.2 and 2.0 the accuracy of nearly all theories does not change very much; only Senthilnathan's theory (S2) has a relative error maximum at $a/b \approx 0.87$. For higher edge ratios it follows Reddy's theory (R3), which is not surprising since the development is based on this theory. With ratios over 2.0 the classical lamination theory (C1) in particular deviates more and more from the 3-D results. To a much smaller degree the same holds also for most of the other plate theories. That can be explained by smaller slenderness ratios in the y-direction. An edge ratio of $a/b = 5$ and a slenderness ratio of $a/h = 15$ lead to a slenderness in the y-direction of $b/h = 3$, which is already quite small. Results obtained with Reddy's theory, however, are hardly affected by the edge ratio variation.

4.3. *Material property relation effect*

Material property relations will certainly have an effect on the deviation from the 3-D results. Especially if the shear modulus is low compared to the in-plane modulus, then the transverse shear effects become more important. For comparison a $[0,90]_s$ laminate is chosen with equal layer thickness and the material properties as listed in Table 5. For $E_L/E_T = 1$ that represents an isotropic material. The edge ratio is determined by eqn (16), thus varying with the material properties between 1.0 and about 1.6.

Slenderness ratios of $a/h = 5$ and 10 are analyzed. Figure 14 shows the center deflections for $a/h = 5$. With increasing E_L/E_T the results increasingly deviate from the 3-D values. The classical laminate theory yields unacceptable deflections even for isotropic material. For high E_L/E_T ratios it is Senthilnathan's theory (S2) which is furthest off.

Already for a slenderness ratio of 10 the deviations from the 3-D results are much smaller as can be seen by comparing Fig. 14 with Fig. 15. But the tendency remains the same. It is worth noting that the theories by Kwon-Akin and Whitney-Pagano seem to be less influenced by a change in the property relation than the others.

5. CONCLUSION

A great number of improved theories for laminated plates has been proposed recently. Information about the necessity and the efficiency of those theories, in which the dis-

Table 5. Material properties for property relation test

$E_L/E_T = 1$ to 50
$E_T/G_{LT} = 2.6$
$\nu_{LT} = 0.3$
$\nu_{TT} = 0.3$

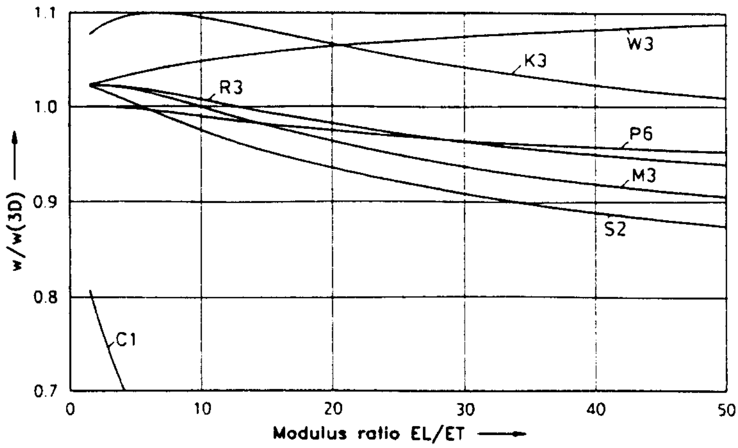

Fig. 14. Center deflection of [0, 90]$_s$ layered plates, $a/h = 5.0$.

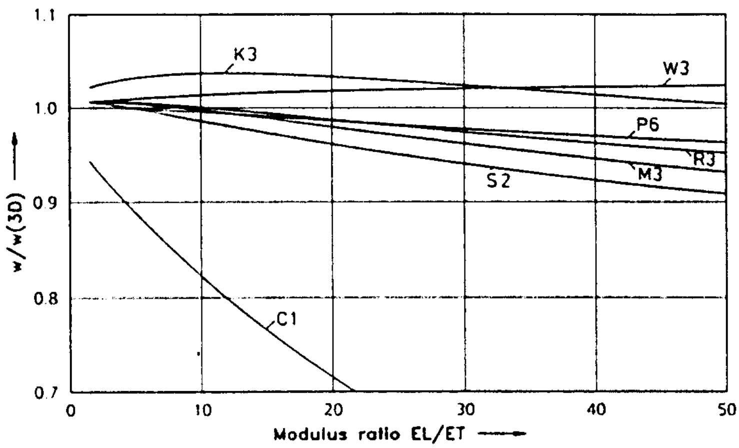

Fig. 15. Center deflection of [0, 90]$_s$ layered plates, $a/h = 10.0$.

placements are approximated by polynomials valid for the total thickness, was the aim of the comparative study presented. It has revealed that for slender plates the classical lamination theory provides satisfying results. Application limits certainly depend on material property relations, but with standard CFRP material a slenderness ratio of $a/h = 25$ led to a center deflection which was already 5% off. For thicker plates a higher order theory is recommended.

One further reason for using a higher order theory is due to the C^1-continuity requirement imposed on trial functions by the Kirchhoff conditions. That makes it difficult to set up simple finite element stiffness matrices. Therefore, quite often the elements are based on the Whitney–Pagano theory which demands C^0 continuity only. However, improved approximations for transverse shear stiffnesses are needed, especially in cases of high modulus ratios. That is not required when applying Murthy's or Reddy's theory, but owing to the second derivatives in the strain–displacement relations they again need C^1-continuous functions.

Considering that it utilizes only two functional degrees of freedom, the theory introduced by Senthilnathan *et al.* has been shown to deliver good transverse displacements in many cases. But the in-plane displacements and the stresses are not very accurate, especially if the material is highly orthotropic with respect to transverse shear. Kwon and Akin's approach yields transverse deflections which are almost always inferior to the Whitney–Pagano theory with improved approximations for transverse shear stiffnesses. More than three functional degrees of freedom, as for instance proposed by Pandya and Kant, are expensive and should be used only when necessary. That would be the case if transverse normal stresses are needed. As a final conclusion it can be stated that Reddy's theory is a good alternative if the classical lamination theory is no longer sufficient. Whitney–Pagano's theory with improved approximations for transverse shear stiffnesses is a good choice if C^1-continuity conditions are to be avoided.

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